Matchings and Covers in bipartite graphs

- A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

- Let $G = (V,E)$ an undirected graph
- A clique is a set of pairwise adjacent vertices (any complete subgraph).
- The problem of finding the maximum size of a clique for a given graph is an NP-complete problem.
- Cliques arise in a number of areas of graph theory and combinatorics, including the theory of error-correcting codes.

- $\omega(G)$ – the number of graph vertices in the largest clique of $G$.
- For the complete graph $K_n$: $\omega(K_n) = n$
- Cubical graph: $\omega(G) = 2$.
- Cycle graph: $\omega(C_n) = 2$ for $n > 3$ and $\omega(C_3) = 3$
- A coclique is a set of pairwise non-adjacent vertices.
- A coclique in a graph is a clique in its complementary graph.
- $\alpha(G) = \max |C| : C$ is a coclique – coclique number

- A vertex cover is a subset $W \subseteq V$ such that $e \cap W \neq \emptyset$ for all $e \in G$.
- The problem of finding the minimum vertex cover for a given graph is an NP-complete problem.
- $\tau(G) = \min |W| : W$ is a vertex cover – vertex cover number.
- Proposition 1: For each $U \subseteq V$ we have: $U$ is coclique if and only if $V \setminus U$ is a vertex cover.

- A matching is a subset $M \subseteq E$ such that: $e \cap e' = \emptyset$ for each $e, e' \in M$.
- The largest possible matching on a graph with $n$ nodes consists of $n/2$ edges, and such a matching is called a perfect matching.
- A perfect matching is a matching which covers all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.
- Although not all graphs have perfect matchings, a maximum matching exists for each graph.
• The maximum matching in a bipartite graph can be found in polynomial time.

\[ \upsilon(G) := \max\{|M| : M \text{ is a matching} \} - \text{matching number.} \]

• An edge cover is a subset \( F \) of \( E \) such that for each vertex \( v \), there exist \( e \in F \) such that \( v \in e \).

Observation: An edge cover can exist only if \( G \) has no isolated vertices.

\[ \rho(G) := \min\{|F| : F \text{ is an edge cover} \} - \text{edge cover number.} \]

• A complete bipartite graph \( G = (V_1, V_2, E) \) is a bipartite graph such that for any two vertices \( v_1 \in V_1 \) and \( v_2 \in V_2 \), \((v_1, v_2)\) is an edge in \( G \). A complete bipartite graph with partitions of size \( |V_1| = m \) and \( |V_2| = n \) is denoted by \( K_{m,n} \).

• A complete bipartite graph \( K_{m,n} \) has a vertex covering number of \( \min\{m, n\} \) and an edge covering number of \( \max\{m, n\} \).

• A complete bipartite graph \( K_{m,n} \) has a perfect matching of size \( \min\{m, n\} \).

• A complete bipartite graph \( K_{m,n} \) has a coclique number of size \( \max\{m, n\} \).

• \( \alpha(G) := \max\{|C| : C \text{ is a coclique} \} - \text{coclique number} \)

• \( \tau(G) := \min\{|W| : W \text{ is a vertex cover} \} - \text{vertex cover number.} \)

• \( \upsilon(G) := \max\{|M| : M \text{ is a matching} \} - \text{matching number.} \)

• \( \rho(G) := \min\{|F| : F \text{ is an edge cover} \} - \text{edge cover number.} \)

Proposition 2: The following inequalities hold:

\[ \alpha(G) \leq \rho(G) \text{ and } \upsilon(G) \leq \tau(G). \]

Observation: Strict inequalities are possible (for example the case of \( C_3 \)).

Theorem (Gallai’s theorem)
For any graph \( G = (V, E) \) without isolated vertices one has:

\[ \alpha(G) + \tau(G) = |V| = \upsilon(G) + \rho(G). \]

Theorem (Konig’s matching theorem)
For any bipartite graph \( G = (V, E) \) one has:

\[ \tau(G) = \upsilon(G). \]

That is, the maximum cardinality of a matching in a bipartite graph is equal to the minimum cardinality of a vertex cover.

Theorem (Konig’s edge cover theorem)
For any bipartite graph \( G = (V, E) \) one has:

\[ \alpha(G) = \rho(G). \]

That is, the maximum cardinality of a coclique in a bipartite graph is equal to the minimum cardinality of an edge cover.

Cardinality bipartite matching algorithm
We focus on the problem of finding a maximum-sized matching in a bipartite graph.

• Let \( M \) be a matching in a graph \( G = (V, E) \).

• A path \( P = (v_0, v_1, \ldots, v_t) \) in \( G \) is called \( M \)-augmenting if

  i) \( t \) is odd and \( v_0, v_1, \ldots, v_t \) are all distinct;

  ii) \( v_1, v_2, \ldots, v_{t-2}, v_{t-1} \in M \);

  iii) \( v_0, v_t \notin M. \)
• If $P = (v_0, v_1, \ldots, v_t)$ is an $M$-augmenting path, then $M' := M \Delta E_P$ is a matching satisfying $|M'| = |M| + 1$.
• In fact, it is not difficult to show that:

Theorem: Let $G = (V, E)$ be a graph and let $M$ be a matching in $G$. Then either $M$ is a matching of maximum cardinality, or there exists an $M$-augmenting path.

So in any graph, if we have an algorithm for finding an $M$-augmenting path for any matching $M$, then we can find a maximum cardinality matching: we iteratively find matchings $M_0, M_1, \ldots$, until we have a matching $M_k$ s.t. there does not exist any $M_k$-augmenting path.

Matching augmenting algorithm for bipartite graphs

Input: a bipartite graph $G = (V, E)$ and a matching $M$,
Output: a matching $M'$ satisfying $|M'| > |M|$ (if there is one),
Description of the algorithm: Let $G$ have colour classes $U$ and $W$. Orient each edge $e = \{u, w\}$ of $G$ (with $u \in U$ and $w \in W$) as follows:
if $e \in M$ then orient $e$ from $w$ to $u$,
if $e \not\in M$ then orient $e$ from $u$ to $w$.
Let $D$ be the directed graph arising in this way. Consider the sets $U' := U \setminus M$ and $W' := W \setminus M$. Now an $M$-augmenting path (if it exists) can be found by finding a directed path in $D$ from any vertex in $U'$ to any vertex in $W'$.
• Hence in this way we can find a matching larger than $M$

Application: The Marriage Problem

Suppose that in a group of $n$ single women and $n$ single men who desire to get married, each participant indicates who among the opposite sex would be acceptable as a potential spouse. This situation could be represented by a bipartite graph in which the vertex classes are the set of $n$ women and the set of $n$ men, and a woman $x$ is joined by an edge to a man $y$ if they like each other.
Could we marry everybody to someone they liked?

• Every woman can be married to at most one man, and every man to at most one woman. Therefore, a possible set of marriages can be represented as a subset $M$ of the edges, no two of which are adjacent (matching).

Thus, the marriage problem can be stated in graph-theoretic terms as asking if a given bipartite graph $G$ has a perfect matching.
Let us suppose that $M$ is a matching, if $M$ is not a maximum matching, how could we improve it by finding a larger one?

So suppose we have the matching Ann married to Bob, Beth to Adam, Christina to Dan, and Fiona to Frank. Dorothy, Evelyn, Carl and Erik are unmatched.

To make progress we must be willing to rearrange our existing matchings, in order to increase their number. But how?

Let us start with a currently unmatched woman, say Dorothy. Now we could reason as follows: to match Dorothy we must marry her to Bob; but Bob is matched to Ann; maybe we could match Ann to someone else; well, we could match Ann to Adam instead, but Adam is already matched to Beth; so if we do that we must match Beth to someone else; we could match Beth to Carl. Carl is currently unmatched so we found a better matching!

The new matching then is Dorothy to Bob, Ann to Adam, Beth to Carl, plus Christina to Dan, and Fiona to Frank who weren't affected by our rearrangement.

We found a way to improve on a matching by finding a path from an unmatched woman to an unmatched man in which every second edge is in the current matching. Such a path is called an $M$-augmenting path.

A vertex coloring is an assignment of labels or colors to each vertex of a graph such that no edge connects two identically colored vertices. The most common type of vertex coloring seeks to minimize the number of colors for a given graph.

The minimum number of colors which with the vertices of a graph $G$ may be colored is called the chromatic number.

It is NP-hard to decide if a graph $G$ is $k$-colourable.

An edge coloring of a graph $G$ is a coloring of the edges of $G$ such that adjacent edges (or the edges bounding different regions) receive different colors.

The edge chromatic number gives the minimum number of colors with which a graph can be colored.

Finding the minimum edge coloring is equivalent to finding the minimum vertex coloring of its line graph.
A line graph $L(G)$ (also called an interchange graph) of a graph $G$ is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if the corresponding edges of $G$ meet at one or both endpoints.

Applications:
- Map colouring;
- Storage of goods;
- Assignment frequencies to radio stations, car phones;
- Scheduling classes.