1 Wahrscheinlichkeitsrechnung II

Probability theory was used in a famous court case: *People vs. Collins* (cf. M.W. Gray, “Statistics and the Law,”, Mathematics Magazine, vol. 56 (1983), pp. 67–81). In this case a purse was snatched from an elderly person in a Los Angeles suburb. A couple seen running from the scene were described as a black man with a beard and a mustache and a blond girl with hair in a ponytail. Witnesses said they drove off in a partly yellow car. Malcolm and Janet Collins were arrested. He was black and though clean shaven when arrested had evidence of recently having had a beard and a mustache. She was blond and usually wore her hair in a ponytail. They drove a partly yellow Lincoln. The prosecution called a professor of mathematics as a witness who suggested that a conservative set of probabilities for the characteristics noted by the witnesses would be as shown below:

man with mustache	1/4
girl with blond hair	1/3
girl with ponytail	1/10
black man with beard	1/10
interracial couple in a car	1/1000
partly yellow car	1/10

The prosecution then argued that the probability that all of these characteristics are met by a randomly chosen couple is the product of the probabilities or 1/12000000, which is very small. He claimed this was proof beyond a reasonable doubt that the defendants were guilty. The jury agreed and handed down a verdict of guilty of second-degree robbery.

Hinweis: People vs. Collins . . . Das Volk gegen Collins; purse . . . Handtasche; to snatch . . . entreissen, stehlen; mustache . . . Oberlippenbart; interracial . . . von unterschiedlicher Hautfarbe; defendant . . . Angeklagter; verdict . . . Schuldspruch.

1. There are two bad mistakes in the argument of the prosecution:

   (a) Is the given probability of 1/12000000 likely to be correct?

   (b) A more devastating, but more subtle argument can be given as follows: Suppose, for example, there are 5000000 couples in the Los Angeles area and the probability that a randomly chosen couple fits the witnesses' description is 1/12000000. Then the probability that there are two such couples given that there is at least one is not at all small. Find this probability. (The California Suppreme Court overturned the initial guilty verdict.)

2 Kontinuierliche Verteilungen

2.1 Exponentialverteilung

2. Zeigen Sie, daß für $\lambda > 0$ die Funktion

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{falls } x \geq 0, \\ 0 & \text{falls } x < 0 \end{cases}$$

eine Dichte ist.

3. Überprüfen Sie die Eigenschaft der "Gedächtnislosigkeit" der Exponentialverteilung: Ist $X \sim Exp(\lambda)$ $(\lambda > 0)$, so ist $P(X > z + x | X > z) = P(X > x)$.

4. Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is the probability that the repair time exceeds 4 hours? If it exceeds 4 hours what is the probability that it exceeds 8 hours?

2.2 Gamma-Verteilung

Für $\alpha \geq 0$ ist die so genannte *Gamma-Funktion* gegeben durch

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

5. Leiten Sie die folgenden Eigenschaften der Γ-Funktion her: $\Gamma(1) = 1$, und $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ für $\alpha \geq 0$. In welcher Beziehung stehen $\Gamma(n)$ und $n!$ für $n \in \mathbb{N}$?

6. Zeigen Sie, daß für $\lambda > 0$, $\alpha > 0$ die Funktion

$$f(x) = \begin{cases} 0 & \text{falls } x < 0, \\ \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{falls } x \geq 0 \end{cases}$$

eine Dichte ist. Die durch diese Dichte beschriebene Verteilung heißt *Gamma-Verteilung mit Parametern* λ *und* α, kurz $Gamma(\lambda, \alpha)$.

7. Prüfen Sie mit Hilfe der entsprechenden Dichtefunktionen, daß gilt:

$$\text{Gamma}(\lambda, 1) \equiv Exp(\lambda).$$

D.h. also, daß die Gamma-Verteilung mit Parametern λ und 1 gleich der Exponentialverteilung mit Parameter λ ist.

2.3 Normalverteilung

8. Sei $X \sim N(\mu, \sigma^2)$ mit $\sigma^2 > 0$. Welcher Verteilung gehorcht $\alpha X + \beta$ ($\alpha, \beta \in \mathbb{R}$)? (Hinweis: Bestimmen Sie mit Hilfe des Transformationssatzes die Dichte von $\alpha X + \beta$.)

9. Es sei $X \sim N(0,1)$, und es sei Φ die dazugehörige Verteilungsfunktion. Zeigen Sie, daß $P(|X| > c) = 2\Phi(-c)$ für $c \geq 0$ ist.

10. Es sei $X \sim N(75, 100)$. Bestimmen Sie $P(X \leq 60)$, $P(70 < X \leq 100)$, sowie c, sodaß $P(-c \leq (X-75)/10 \leq c) = 0.95$.